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Gauge invariance in second-class constrained systems

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Abstract. We show that many second-class constrained Hamiltonian systems can be reformulated as first-class systems within the same phase space.

Dirac [1] classified the constraints of a Hamiltonian system as being first and second class, the former having vanishing Poisson brackets (PB) constraint algebra on the constrained surface, and the latter non-vanishing. Second-class systems naturally define a local projected symplectic manifold, while the first-class constraints imply existence of redundant so-called gauge degrees of freedom; consequently the Hamiltonian dynamics has gauge invariance.

Symmetries of a system are useful in understanding the dynamics. If we reformulate a second-class system as first class we are exhibiting gauge symmetries, namely the ones associated with the first-class constraints, at the kinematic level. Recently gauge invariance and its associated symmetries have become an important feature in establishing the existence of the corresponding quantum theory, noted examples being renormalizability of QED and QCD. There are other examples [2], such as anomalous gauge theories, wherein classical gauge symmetries cease to exist upon quantization. The existence of these quantum theories are plagued with problems of renormalizability. It is believed that these issues can be settled by understanding the underlying symmetries in the quantum theory.

Faddeev and Shatashvili [3] have argued a simple general principle by which one enlarges the phase space and introduces a compensatory dynamics such that in the total phase space one realizes only first-class constraints. This has been adopted in the BRST framework by Batalin and Fradkin [4]. Mitra and Rajaraman [5] found that in certain dynamical systems the reformulation of second-class constraints as first class may possibly be done without enlarging the phase space. They showed explicitly that in a special class of Lagrangian dynamics, which yields hierarchical constraint PB algebra, one can eliminate all second-class constraints and obtain only first-class constraints. In this paper we will show that many Hamiltonian systems (irrespective of the existence of a Lagrangian) with second-class constraints can be reformulated as first-class constrained systems. There are some which are essentially second class, namely they do not admit gauge-like constraints globally although they do admit them locally.

In a first-class constrained system with Hamiltonian H and one constraint $\chi = 0$, by making the gauge fixing choice $\psi = 0$ and suitably modifying the Hamiltonian we can define a second-class system with the same physical content as the original system [6].

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In constrast, we consider here a system with two second-class constraints and identify one constraint as a gauge-fixing-like constraint $\psi = 0$ and call the other $\chi = 0$ the first-class-like constraint. Although naively true, in general this poses a problem, namely the constraint $\chi = 0$ will be preserved in time only modulo the constraint $\psi = 0$. To avoid this technical difficulty we construct a modified Hamiltonian $\widetilde{H}$ which preserves the constraint $\chi = 0$ in time evolution, such that if one makes the gauge fixing choice $\psi = 0$, we should have $\widetilde{H} = H$, thus ensuring that the physical content of the new dynamics is equivalent to our original dynamics.

For simplicity we consider first the case where there is only one pair of second-class constraints. Then we show how we can treat the general case. Examples of the application of this algorithm will be considered in a separate publication.

Consider a Hamiltonian $H(p, q)$ generating the dynamics over a phase space (p, q) with canonical PB algebra. For brevity we shall not write a distinguishing index for the various p's and q's explicitly. Let the system have two second-class constraint functionals $\chi(p, q) = 0$ and $\psi(p, q) = 0$ with the following PB relation:

$$\{\chi, \psi\} = E \tag{1}$$

where $E \neq 0$ on the surface defined by the two constraints. Without losing any generality we can rewrite (1) as

$$\{\chi', \psi\} = 1 + \{E^{-1}, \psi\}E\chi' \tag{2}$$

where $\chi' \equiv E^{-1}\chi$.

In general, since the constraints $\chi = 0$ and $\psi = 0$ are preserved under the evolution by H on the surface defined by both the constraints, we have

$$\{\chi, H\} \cong a\psi \tag{3}$$

where we have omitted the superscript on χ and '$\cong$' implies equality on the surface defined by only $\chi = 0$. Also $a(p, q)$ does not necessarily vanish.

Our task now is to modify H such that $\chi = 0$ becomes time-independent modulo only $\chi = 0$. On the symplectic manifold let us work on a patch near a zero of the functional $\psi(p, q)$. By local canonical transformation we can indeed take χ and ψ itself as one pair of approximate canonical co-ordinates, due to (2). If we make a Taylor expansion of H in the variable ψ, the leading term H_0 will have vanishing PB with χ by definition. Furthermore on the constrained surface $\psi = 0$, we get $H = H_0$. Hence, locally H_0 is our desired modified Hamiltonian. Implementing this idea analytically we have a prescription for constructing a global analogue of H_0 from H by the use of the following Lie projection operator $\mathbb{P}$: for any arbitrary functional $A(p, q)$ on the phase space

$$\begin{aligned} \widetilde{A} &\equiv \mathbb{P}A(p, q) \\ &\equiv \sum_{n=1}^{\infty} \frac{1}{n!}(-\psi)^n(\hat{\chi})^n A \\ &\equiv :\mathrm{e}^{-\psi\hat{\chi}}: A \end{aligned} \tag{4}$$

where $\hat{\chi}$ is a Lie operator [6] defined by

$$\hat{\chi}A \equiv \{\chi, A\}. \tag{5}$$

In (4) the last expression defines our normal ordering of the Lie operator, namely all the functionals ψ are to be placed on the left in the Taylor expansion. The projection operator is defined globally on the phase space, namely a power series expansion always exists for any $\widetilde{A}$.

The projection operator $\mathbb{P}$ acts linearly on any functional defined on the phase space and satisfies the following relations on the constrained surface $\chi = 0$:

$$\mathbb{P}^2 \cong \mathbb{P} \tag{6a}$$

$$\hat{\chi}\mathbb{P} \cong 0 \tag{6b}$$

$$\mathbb{P}\psi \cong 0 \tag{6c}$$

$$\{\widetilde{A}, \widetilde{B}\} \cong \mathbb{P}(\{A, B\} - \{\psi, A\}\{B, \chi\} + \{\psi, B\}\{A, \chi\}) \tag{7}$$

where the right-hand side of (7) is the Lie projection on the Dirac bracket between A and B.

The projected functionals also satisfy the Jacobi identity on the constrained surface, namely

$$\{\widetilde{A}, \{\widetilde{B}, \widetilde{C}\}\} + \{\widetilde{B}, \{\widetilde{C}, \widetilde{A}\}\} + \{\widetilde{C}, \{\widetilde{A}, \widetilde{B}\}\} \cong 0\,. \tag{8}$$

Furthermore

$$\mathbb{P}(AB) = (\mathbb{P}A)(\mathbb{P}B) = \widetilde{A}\widetilde{B} \tag{9a}$$

and

$$\widetilde{A}(p, q) = A(\widetilde{p}, \widetilde{q})\,. \tag{9b}$$

Applying the projection operator on H, we obtain $\widetilde{H}$ such that $\widetilde{H} \cong H_0$, i.e. $\widetilde{H}$ differs from H_0 by terms proportional to χ. By construction we have $\{\chi, \widetilde{H}\} \cong 0$. Hence our first-class dynamical system is defined by the constraint $\chi \cong 0$ and the Hamiltonian $\widetilde{H}$.

A few remarks are in order about our modified Hamiltonian $\widetilde{H}$. H_0 is defined locally, hence the equality $\widetilde{H} \cong H_0$ is to be understood locally. However, the left-hand side $\widetilde{H}$ is defined globally by construction; consequently we have our desired result globally. It is quite clear that a global H_0 may be impossible. For example, if $\psi = 0$ were a polynomial equation of degree greater than one in p and q, then by a suitable choice we may solve for some phase space variable explicitly. But we cannot eliminate it for we have more than one permitted solution. Whenever we have more than one permitted solution we have to analyse within the patch of a solution thus restricting ourselves to local analysis. But it is interesting that in any patch we can add terms proportional to χ such that the resultant function $\widetilde{H}$ is globally defined. In other words H_0 in one patch can be matched with H_0 in another patch upto gauge transformations.

To illustrate our formalism, we consider a free particle moving along two intersecting circles with centres at $\pm\boldsymbol{a}$ of radius R and given by the constraint

$$\begin{aligned} Q_1 &= [(\boldsymbol{r} + \boldsymbol{a})^2 - \boldsymbol{R}^2][(\boldsymbol{r} - \boldsymbol{a})^2 - \boldsymbol{R}^2] \\ &= (\boldsymbol{r}^2 + \boldsymbol{a}^2 - \boldsymbol{R}^2)^2 - 4(\boldsymbol{a} \cdot \boldsymbol{r})^2 = 0\,. \end{aligned} \tag{10}$$

The Hamiltonian is

$$H = \frac{1}{2m}\boldsymbol{p}^2\,. \tag{11}$$

The constraint (10) is preserved in time if we further impose the constraint

$$Q_2 = \frac{4}{m}[\boldsymbol{r} \cdot \boldsymbol{p}(\boldsymbol{r}^2 + \boldsymbol{a}^2 - \boldsymbol{R}^2) - 2(\boldsymbol{a} \cdot \boldsymbol{p})(\boldsymbol{r} \cdot \boldsymbol{a})] = 0\,. \tag{12}$$

These constraints form the Poisson bracket algebra

$$\{Q_1, Q_2\} = \frac{64}{m}(\boldsymbol{a}\cdot\boldsymbol{r})^2\boldsymbol{R}^2 + \frac{16}{m}\boldsymbol{r}^2 Q_1 \equiv E\,. \tag{13}$$

Note that E is non-zero everywhere except at the points of intersection. Thus (10) and (12) form a second-class constrained system.

To carry out the gauge reformulation of the above theory, we redefine the constraints as

$$\chi = E^{-1}Q_1 \qquad \psi = Q_2\,. \tag{14}$$

Defining the projection operator of the type (4), we get our gauge invariant Hamiltonian on the surface given by (10)

$$\begin{aligned}\widetilde{H} &= \mathbb{P}H = H - \frac{1}{2}E^{-1}\psi^2 = \frac{1}{2m}p_i\overline{\delta}_{ij}p_j\\ \overline{\delta}_{ij} &= \delta_{ij} - \frac{64}{mE}\left[(\boldsymbol{a}\cdot\boldsymbol{r})^2(a_ia_j + r_ir_j) - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{r})(\boldsymbol{r}^2 + \boldsymbol{a}^2 - \boldsymbol{R}^2)(a_ir_j + a_jr_i)\right].\end{aligned} \tag{15}$$

The gauge invariant Hamiltonian $\widetilde{H}$ is invariant under the transformation

$$\boldsymbol{p} \to \boldsymbol{p} + \frac{4\mu}{E}[(\boldsymbol{r}^2 + \boldsymbol{a}^2 - \boldsymbol{R}^2)\boldsymbol{r} - 2(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{a}]$$

where μ is the arbitrary gauge transformation parameter.

We notice that the metric $\overline{\delta}$ is singular on the constraint surface, i.e $\det\overline{\delta} = 0$ on (10). Furthermore $\det\overline{\delta}$ is not of any definite sign on the configuration space. The Ricci scalar curvature is singular on (10) as $\chi^{-5/2}$.

We remark that if we interchange the roles of Q_1 and Q_2 in (14), we get a Hamiltonian $\widetilde{H}$ which is an infinite series.

Next we consider the case where there are finitely many constraints $\phi_i(p,q) = 0$, $i = 1, 2, \ldots, 2N$ defining the constraint surface $\sum_{2N}$, with the PB algebra

$$\{\phi_i, \phi_j\} = E_{ij} \tag{16}$$

where E is globally invertible on $\sum_{2N}$. Our first task is to classify these constraints into gauge-generator-like constraints χ and gauge-fixing-like constraints ψ. Locally on the phase space there are indeed many choices available since, due to Darboux's theorem, there are linear combinations of the ϕ_i which have an E matrix in a Jordan canonical form. However, for our purpose here we need to be able to do this globally. In general we do not have a global analogue of Darboux's theorem. Although in many explicit examples in field theory this is not a serious problem, we shall look into it in detail. We find that whenever the E matrix has a submatrix of dimension N which is globally invertible on $\sum_{2N}$ then there exist gauge-generating-like and gauge-fixing-like constraints globally. To see this let us assume we can write E as

$$E = \begin{pmatrix} E_1 & E_3 \\ -E_3^{\mathrm{T}} & E_2 \end{pmatrix} \tag{17}$$

where E_1, E_2 and E_3 are matrices of dimension N and for a suitable choice of index labelling we can assume that E_3 is invertible on $\sum_{2N}$.

Making a transformation on the ϕ_i to $\phi'_i = A_{ij}\phi_j$ where the $2N$-dimensional matrix A is given by (18) below we can have our new matrix E' as in (18). On the surface $\sum_{2N}$

$$\begin{pmatrix} 1 & A_3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_1 & E_3 \\ -E_3^{\mathrm{T}} & E_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ A_3^{\mathrm{T}} & 1 \end{pmatrix}=\begin{pmatrix} 0 & E_3' \\ -E_3'^{\mathrm{T}} & E_2 \end{pmatrix} \tag{18}$$

where A_3 satisfies the equation

$$A_3E_2A_3^{\mathrm{T}}+E_3A_3^{\mathrm{T}}-A_3E_3^{\mathrm{T}}+E_1=0\,. \tag{19}$$

The N dimensional matrix A_3 can be solved as a series in E_1, since E_3 is invertible. This solution is global. Then our gauge generators are $\chi_a \equiv \phi_a'$ for $a=1,2,\ldots,N$ and $\psi_a \equiv \phi_{N+a}'$ such that $\{\chi_a,\psi_b\}=(E_3')_{ab}$ and $\{\chi_a,\chi_b\}=0$ on $\sum_{2N}$.

From our construction, E_3' is also invertible since $\det E \neq 0$. Therefore by redefining χ ($\chi' = E_3^{-1}\chi$) we have on the surface $\sum_{2N}$

$$\{\chi_a,\psi_b\}=\delta_{ab} \tag{20a}$$

and

$$\{\chi_a,\chi_b\}=0 \tag{20b}$$

for $a=1,2,\ldots,N$.

Now we have to describe a consistent dynamics on the surface defined by $\chi_a=0$ alone. We follow a hierarchical scheme. By suitable rescaling of χ_1 we have

$$\{\chi_1,\psi_1\}\cong_1 1$$

where '$\cong_1$' implies equality on the surface $\chi_1=0$ alone. We construct the projection operator

$$\mathbb{P}_1 =: \exp\left(-\psi_1\widehat{\chi}_1\right):\,.$$

The PB of χ_1 with the projected functionals $\widetilde{H}^1$ and $\widetilde{\chi}_a{}^1,\widetilde{\psi}_a{}^1$ vanishes on the $\chi_1=0$ surface. Furthermore on the surface defined by $\chi_1=0$ and $\widetilde{\chi}_a{}^1=0=\widetilde{\psi}_a{}^1$ for $a=2,3,\ldots,N$, we have

$$\{\widetilde{\chi}_a{}^1,\widetilde{\psi}_b{}^1\}=\delta_{ab} \tag{21a}$$

$$\{\widetilde{\chi}_a{}^1,\widetilde{\chi}_b{}^1\}=0\,. \tag{21b}$$

Consequently, χ_1 is a proper first-class constraint of our dynamical system with the other constraints forming a second-class algebra. By suitably rescaling $\widetilde{\chi}_2{}^1$ we can arrange that $\{\widetilde{\chi}_2{}^1,\widetilde{\psi}_2{}^1\}\cong_2 1$. We then define

$$\mathbb{P}_2 =: \exp\left(-\widetilde{\psi}_2{}^1\widehat{\widetilde{\chi}}_2{}^1\right):\mathbb{P}_1 \tag{22}$$

and get $\sim_2$ functionals which are consistent with $\widetilde{\chi}_2{}^1=0$ and $\chi_1=0$ as being first-class constraints. Iterating this procedure, we define

$$\mathbb{P}_a =: \exp\left(-\widetilde{\psi}_a{}^{a-1}\widehat{\widetilde{\chi}}_a{}^{a-1}\right):\mathbb{P}_{a-1}\,.$$

The projection operator $\mathbb{P}_N$ makes $\widetilde{H}^N$ as the globally defined Hamiltonian equivalent to our original dynamical system and $\widetilde{\chi}_a{}^{a-1}=0$ as our first-class constraints. In general these generators obey the following PB algebra: for $b<a$

$$\{\widetilde{\chi}_a{}^{a-1},\widetilde{\chi}_b{}^{b-1}\}=g_{ab}\widetilde{\chi}_b{}^{b-1}\,. \tag{23}$$

A few remarks are in order. It is evident that there can be second-class systems, whose E matrix (17) has a globally invertible submatrix E_3 of dimension $n < N$. In this case from our constructive proof given above we will have only n gauge globally definable generators. Having additional first-class constraints along with second-class ones to begin with does not hinder our construction of this projection operator. We find that in general, any constrained system with maximal number of constraints, say n, whose PB algebra vanishes globally, admits n gauge generator-like constraints. To reiterate there are certain dynamical systems which are essentially second class in the sense they do not admit gauge-like constraints globally. These need to be handled by the Dirac bracket formalism.

The first-class constraints χ_i define the gauge generators. In our algorithm we find that χ_i are not uniquely defined, i.e. the gauge generators are ambiguous (we saw this in the example given earlier). The ambiguity is lifted if we pay more attention to our projection operator. Although $\mathbb{P}$ is formally well defined on any functional $A(p, q)$ as a power series, the series may or may not converge. More specifically if $\widetilde{A}(p, q)$ needs to be a physical variable the convergence of this series is necessary. We find in explicit examples that one choice of χ yields in fact a finite series while an alternative choice yields an infinite series. Consequently, we believe that in general there are other considerations which make the choice of χ unique, hence the gauge group. This shall be discussed in detail in a later publication [7]. We have applied our method to various cases; in particular those considered in [5] are also examples of our method.

We make a comment about a second-class constrained system which has a subset of constraints χ_a which form a first-class PB algebra within themselves, namely $\{\chi_a, \chi_b\} = f_{abc}\chi_c$. In these cases we need not go through the hierarchical process explained above. Here we only have the problem of getting the modified Hamiltonian which ensures the time-independence of the first-class subset of constraints. The alternative algorithm stated in [7] yields a modified Hamiltonian which is again a series in ψ_i, the set of gauge fixing constraints corresponding to the first-class constraints. It is interesting that in this case the structure functions of the constraint PB algebra, i.e. the analogues of g_{abc} vanish on the first-class constrained surface.

All our discussion has been classical. The canonical quantization program can be envisaged quite naturally with the projected variables. An important requirement being the Jacobi identity (9) is satisfied on the constraint surface. The standard operator ordering problems which occur when we go from classical to quantum variables have to be reconsidered. Alternatively we may think of defining the projection operator on the Hilbert space directly, i.e. define the operator $\hat{\chi}$ on the Hilbert space as a commutator $\hat{\chi}A = [\chi, A]$. In general this may not yield a projection operator due to ordering problems in the series in (4). In either case on a finite dimensional phase space, we can always find an operator ordering satisfying the desired property of the projection operator [8].

Our method is applicable to classical field theories with second-class constraints. Upon quantization of these field theories we cannot guarantee that there will not be any anomalies. This is a subject matter of great interest which is being looked into. In the literature [9] there has been an interesting application of removing quantum anomalies in a semiclassical sense using this type of construction.

Formally speaking our construction of the projection operator is also amenable to fermionic constraints, such as those in BRST quantization. All Poisson brackets [4] have to be interpreted as anti-commutators and some attention should be given to the orderings of the Grassmann variables.

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